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LETTER TO THE EDITOR

Fluctuations of the Lyapunov exponent in disordered systems

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Abstract. The fluctuations of the maximum Lyapunov exponent of a product of random matrices are studied analytically and numerically. It is shown that they may be expressed as a sum of two terms, one related to the order of matrices within the product, the other to fluctuations of the number of matrices of a given type. This result is then applied to the one-dimensional random-field Ising model and the discrete Schrödinger equation with a random potential.

The Lyapunov characteristic exponents are very useful for characterising the properties of dynamical as well as disordered systems. Their significance becomes especially transparent in the transfer matrix formalism. In spin systems, for example, the maximum Lyapunov exponent is related to the free energy; for the Schrödinger equation in a random potential, the minimum non-negative Lyapunov exponent is the inverse localisation length [1, 2].

The purpose of this paper is to discuss the problem of finite-volume (for disordered systems) or finite-time (for dynamical systems) fluctuations of the Lyapunov exponent [3-5]. It will be shown that these are made up of two terms: one related to the ordering of the matrices within the product, the other to the fluctuations of the number of each type. This result will then be applied to two cases of physical interest, namely the one-dimensional Ising model in a random magnetic field and the one-dimensional discrete Schrödinger equation in a random potential.

Let us recall some basic definitions to set notation and terminology. The maximum *Lyapunov characteristic exponent* (LCE in the following) or a product of N, $d \times d$, independent random matrices M_i is defined through the following relations:

$$G_N = \prod_{i=1}^{N} M_i$$
$$\lambda = \lim_{N\to\infty} \frac{1}{N} \ln\left(\frac{\|G_N z(0)\|}{\|z(0)\|}\right) \tag{1}$$

where $z(0)$ is a generic vector in R^d. The LCE is a non-random quantity, in the sense that it does not depend on the particular realisation of G_N [6]. Nevertheless, it is subject to finite-N fluctuations around its asymptotic value. The probability $P_N(\gamma)\,d\gamma$,

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that the effective Lyapunov exponent, for a system of size N, assumes a value between γ and $\gamma+d\gamma$, may be reconstructed through the 'generalised Lyapunov exponents' [5, 7]:

$$L(Q)=\lim_{N\to\infty}\frac{1}{N}\ln\left\langle\left(\frac{\|G_N z(0)\|}{\|z(0)\|}\right)^Q\right\rangle \tag{2}$$

where the average is taken over all possible realisations of G_N. From standard probability theory $L(Q)/Q$ is a non-decreasing function of Q. We will deal with two 'ensembles' in the following, the *canonical*, where the number of each type of matrices is allowed to vary and the *microcanonical*, denoted by $\langle\cdot\rangle_M$, where it is kept fixed. Let us stress that the LCE is the same in both ensembles since, for almost all realisations, there is a unique limit of (1) [6]. This is not true for the moments as one can easily check in the case of a product of *commuting* random variables.

From now on we consider Bernoulli trials, i.e. $M_i=A$ with probability p and $M_i=B$ with probability $q=1-p$. However, everything may be extended to more complicated cases.

We have recently estimated the LCE in this case [8] by computing the so-called annealed average $\langle G_N\rangle_M$ in the microcanonical ensemble. In fact, by the binomial formula, one sees that

$$\langle G_N\rangle_M=\binom{N}{qN}^{-1}\frac{1}{(qN)!}\frac{\partial^{qN}}{\partial x^{qN}}(A+xB)^N|_{x=0} \tag{3}$$

since the derivatives kill the realisations with too many As and taking $x=0$ kills those with too many Bs. Cauchy's theorem leads to

$$\langle G_N\rangle_M=\binom{N}{qN}^{-1}\frac{1}{2\pi i}\int_\Gamma\frac{l^N(z)D(z)}{z^{qN+1}}\,dz \tag{4}$$

where $l(x)$ is the largest eigenvalue of the matrix $A+xB$, $D(x)$ is a $d\times d$ matrix, whose elements are $O(1)$ and Γ is a contour in the complex plane around $z=0$. In the large-N limit the integral may be evaluated by the saddle point method

$$\langle G_N\rangle_M\simeq\exp N[p\ln p+q\ln q+\ln\tilde{l}-q\ln\tilde{z}] \tag{5}$$

where $\tilde{l}=l(\tilde{z})$ refers to the saddle point $z=\tilde{z}$, solution of

$$z\frac{d\ln l(z)}{dz}=q. \tag{6}$$

We can thus estimate the LCE by the microcanonical annealed average:

$$\tilde{L}(1)=\mathrm{Re}(p\ln p+q\ln q+\ln\tilde{l}-q\ln\tilde{z}) \tag{7}$$

which has been found to work quite well in many cases [8] (beyond the trivial one of commuting or upper (lower) triangular matrices A and B, where one has $\tilde{L}(1)=\lambda$). We must stress that $\tilde{L}(1)$ is different from $L(1)$ defined by (2) if the matrix M_i has some negative elements. $\tilde{L}(1)$ can be larger or smaller than λ, while $L(1)\geqslant\lambda$. It is also possible to calculate the other (integer) moments $\langle G_N^Q\rangle_M$ by noting [9] that

$$\left\langle\left(\mathrm{Tr}\prod_{i=1}^N M_i\right)^Q\right\rangle=\left\langle\mathrm{Tr}\prod_{i=1}^N M_i^{\otimes Q}\right\rangle \tag{8}$$

where $M_i^{\otimes Q}=M_i\otimes\ldots\otimes M_i$ (Q times the usual tensor product). This allows us to use the same trick, equation (3), replacing A and B with $A^{\otimes Q}$, $B^{\otimes Q}$ respectively. The computation is quite non-trivial but may still be carried through.

It is now possible to expand the generalised Lyapunov exponents in Taylor series around $Q=0$

$$L(Q)=\lambda Q+\frac{\mu}{2}Q^2+O(Q^3) \tag{9}$$

where μ (called the 'non-uniformity factor' (NUF) in [3]) is given by

$$\mu=\lim_{N\to\infty}\frac{1}{N}\left[\left\langle\ln^2\left(\frac{\|G_N z(0)\|}{\|z(0)\|}\right)\right\rangle-\lambda^2N^2\right]. \tag{10}$$

Let us now derive a simple formula for computing μ. In the microcanonical ensemble we can also introduce a new set of generalised Lyapunov exponents

$$L_f^*(Q)=\lambda^*(f)Q+\frac{\mu^*(f)}{2}Q^2+O(Q^3) \tag{11}$$

where μ^* is the variance due to matrix ordering in G_N at fixed number, fN, of type-A matrices. The two ensembles are related through standard arguments of statistical mechanics so that for any given p:

$$\left\langle\left(\frac{\|G_N z(0)\|}{\|z(0)\|}\right)^Q\right\rangle\propto\int e^{L_f^*(Q)N}\mathscr{P}_N(f)\,df \tag{12}$$

where $\mathscr{P}_N(f)$ is the probability that, in a product G_N, there are fN type-A matrices. It may be approximated by a Gaussian, $\mathscr{P}_N(f)\simeq\exp-N(f-p)^2/(2\sigma^2)$, for large N and in the absence of correlations. Calculating the above integral by the saddle point method, we can relate $L_f^*(Q)$ to $L(Q)$ within the canonical ensemble:

$$L(Q)=\max_f\left\{L_f^*(Q)-\frac{(f-p)^2}{2\sigma^2}\right\} \tag{13}$$

where, in our case, σ is the variance of the binomial distribution, $\sigma^2=p(1-p)$. For small Q, we obtain for the frequency $\bar{f}$ that realises the maximum in (13):

$$\left.\frac{d\lambda}{df}\right|_{\bar{f}}Q\sigma^2+p=\bar{f} \tag{14}$$

and the generalised exponents

$$\begin{aligned}L(Q)&=L_{\bar{f}}^*(Q)-\frac{(\bar{f}-p)^2}{2\sigma^2}\\&=\lambda^*(p)Q+\frac{1}{2}\left\{\mu^*(p)+\left(\left.\frac{d\lambda}{df}\right|_{f=p}\right)^2\sigma^2\right\}Q^2+O(Q^3).\end{aligned}$$

This leads to an explicit formula for μ:

$$\mu=\mu^*+\sigma^2\left(\frac{d\lambda}{df}\right)^2_{f=p}. \tag{15}$$

If the microcanonical fluctuations are absent, then ($\mu^*=0$) and the microcanonical estimates give the exact results for both λ and μ. Otherwise a rough estimate of μ^* is provided by an analytical calculation of $L^*(2)-2\tilde{L}(1)$, via the microcanonical trick.

We have explicitly carried out the above computations in the case of the random-field Ising model and the discrete Schrödinger equation in a random potential and compared them with numerical simulations.

The model is defined through the Hamiltonian

$$\mathcal{H} = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h_i \sigma_i \tag{16}$$

where $h_i = h$ with probability p and $h_i = 0$ otherwise. The transfer matrix may be defined in the following way [1]:

$$T_i = \exp \beta [J\sigma_i \sigma_{i+1} + h_i \sigma_i]$$
$$= e^{\beta(J+h_i)} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon \cdot z_i & z_i \end{pmatrix}$$

where $\varepsilon = e^{-2\beta J}$, $z_i = e^{-2\beta h_i}$. The non-trivial contribution to the free energy per spin, i.e. Lyapunov exponent, is given by the product of matrices that may take one of two possible forms

$$A = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon \cdot z & z \end{pmatrix} \tag{17}$$

with probability p and $1-p = q$ respectively. Here $z = e^{-2\beta h}$ and $J = 1$ (ferromagnetic case). We have rescaled the variables so that the two values of the field are h and zero. The independent parameters are the temperature, the magnetic field and the value of p.

The explicit calculation proceeds as follows: we calculate the eigenvalues of the matrix $A + xB$; this involves solving a quadratic equation. These are then input for the saddle-point equation (6), which ends up as a fourth-order algebraic equation. This may be easily solved by standard numerical techniques, or analytically. We thus arrive at four possible candidates for $\tilde{l}$ and, consequently, for $\tilde{L}(1) = L^*(1)$, since these matrices have positive elements. Since the eigenvalue of $\langle G_N \rangle_M$ is related to the free energy, it must be real, which leads to a unique answer. As regards $L^*(2)$, the computation is slightly more involved. Within the microcanonical ensemble, one must first find the eigenvalues of the 4×4 matrix $A \otimes A + xB \otimes B$, i.e. solve a fourth-order equation. This may actually be done analytically, since it is found that the characteristic polynomial factors into a linear and a cubic piece (this factorisation is a general property of the characteristic polynomial of direct products of matrices, according to which the computation of the roots of the d^Q-degree polynomial of a direct product of Q matrices reduces to that of a polynomial or order $Q+1$; in our case, this reduction leads from a fourth-degree to a cubic polynomial [9, 10]). This was realised with the aid of an algebraic manipulation program. The saddle-point equation (6) is then set up. Care must be taken to find (numerically) the correct saddle point in the complex plane, since it appears that the number of 'spurious' ones is quite large.

Within the *canonical* ensemble, on the other hand, calculating $L(2)$ is much simpler [9]:

$$L(2) = \ln(\max \text{ eigenvalue } (pA \otimes A + (1-p) B \otimes B)). \tag{18}$$

In figure 1 we compare $\tilde{L}(1)$ with λ. We see that the agreement is very good, indicating that the principal source of fluctuations is the second term of (15). Indeed, we have checked that also $L^*(2)/2$ is very close to λ. The fact that the ordering fluctuations

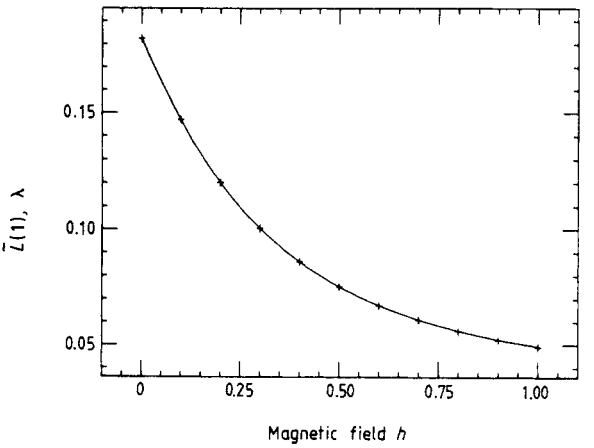

Figure 1. $\tilde{L}(1)$ (full curve) and λ (+) against h for $p = 0.5$ and $\varepsilon = 0.2$.

are negligible is strengthened by figure 2, where we present a numerical calculation of μ compared with an (analytical) calculation of $p(1-p)(\mathrm{d}\tilde{L}(1)/\mathrm{d}p)^2$. Moreover, we have done a numerical calculation of the microcanonical variance μ^* and found that for a wide range of temperatures $\mu^* \simeq 10^{-6}$, that is 0.1% of μ.

We turn now to the Schrödinger equation in a random potential. The physical problem consists in the properties of the electronic wavefunctions in a one-dimensional lattice, on each site of which the potential V_i is a random variable [1, 2]. We consider the case of a binary alloy, i.e. $V_i = V_a$ with probability p, and $V_i = V_b$ with probability $q = 1 - p$. The Schrödinger equation has the standard form $\psi_{i+1} - 2\psi_i + \psi_{i-1} + V_i\psi_i = E\psi_i$ and may be written in terms of transfer matrices as

$$\Psi_{i+1} = T_i\Psi_i \qquad \Psi_i = \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix} \tag{19}$$

where T may take two values

$$A = \begin{pmatrix} E - V_a + 2 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} E - V_b + 2 & -1 \\ 1 & 0 \end{pmatrix}$$

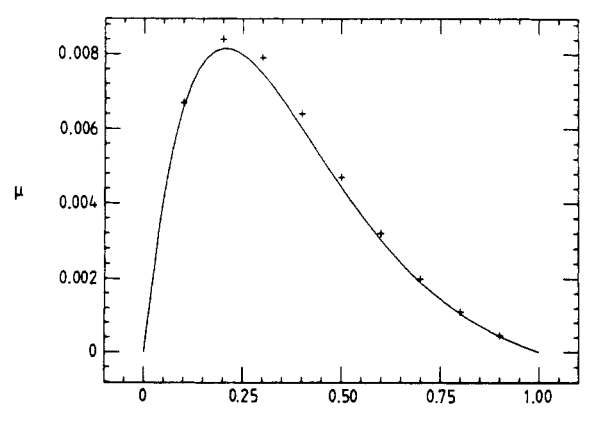

Figure 2. Variance μ (+) as a function of p for $h = 0.5$ and $\varepsilon = 0.2$. The full curve is $p(1-p)(\mathrm{d}\tilde{L}(1)/\mathrm{d}p)^2$.

with probability p and $1-p=q$ respectively. Depending on the value of the energy E, the eigenvalues of these matrices are real or complex. They are complex within the range

$$V_a - 4 \leqslant E \leqslant V_a \qquad \text{or} \qquad V_b - 4 \leqslant E \leqslant V_b. \tag{20}$$

Let us recall that the LCE is the inverse of the localisation length while the imaginary part of the logarithm of the eigenvalue of G_N is related to the integrated density of states.

In our calculations we have taken $E=-2$. Figure 3 displays the exponents $\tilde{L}(1)$, $L^*(2)/2$ (computed by the microcanonical trick) and λ as a function of $\Delta V = 1.9 - V_b$, for $p=0.5$. It is quite impressive that $\tilde{L}(1)$ approximates λ pretty well, although the variance μ is very large ($\mu \simeq \lambda$).

We investigated in detail the most unfavourable case of figure 3, i.e. $V_b = 0$. In figure 4, we show $\tilde{L}(1)$ and $L^*(2)/2$ compared with λ, as function of p. In figure 5 we present μ and μ^* in the same case. We notice that the microcanonical fluctuations

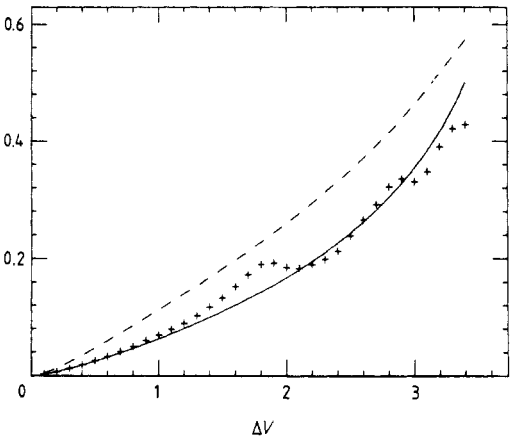

Figure 3. $\tilde{L}(1)$ (full curve), $L^*(2)/2$ (broken curve) and λ (+) against $\Delta V = V_b - 1.9$, for $p=0.5$.

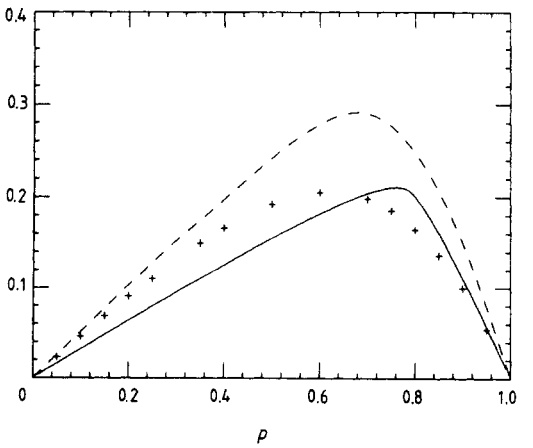

Figure 4. $\tilde{L}(1)$ (full curve) and $L^*(2)/2$ (broken curve) compared with λ, against p, for $V_a = 1.9$ and $V_b = 0$.

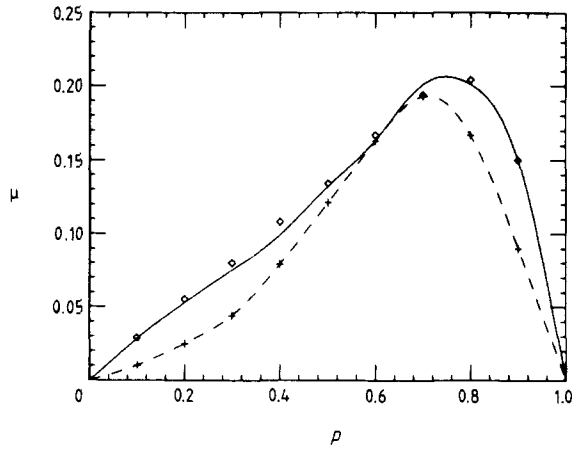

Figure 5. μ ($\diamond$) and μ^* (+) against p for $V_a = 1.9$ and $V_b = 0$. The full curve is the sum of μ^* and $p(1-p)(d\lambda/dp)^2$. We have drawn a broken curve interpolating the numerical values of μ^* only for clarity.

give the leading contribution to μ, in contrast to the Ising model. The full curve represents the sum of the numerical results for μ^* and $p(1-p)(d\lambda/dp)^2$, which is in good agreement with the numerical values of μ.

To summarise, we have obtained an explicit expression for the fluctuations of the Lyapunov exponent around its asymptotic value within the microcanonical and canonical ensembles and have used it to study two physically interesting cases, the random-field Ising ferromagnet and the discrete Schrödinger equation with a random potential, in one dimension. In the first case the main contribution comes from the number fluctuations whereas, in the localisation problem, the ordering fluctuations seem to dominate. A fuller study of these issues will be reported elsewhere.

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